

Coordinate Transformation

Interchanging variables and operators in spherical and cylindrical coordinate system from cartesian

revised on 2026-06-02 | written by Raunak Farhaz

Polar or Spherical Coordinate System

The spherical or polar coordinate system takes into account one radial component and two angular component to span the space. They are

$$\begin{aligned} r &\rightarrow \text{radial component} \\ \theta &\rightarrow \text{polar angle} \quad [0 \leq \theta \leq \pi] \\ \phi &\rightarrow \text{azimuthal angle} \quad [0 \leq \phi \leq 2\pi] \end{aligned} \quad (1)$$

It can be illustrated which of the parameter does what in the following diagram.

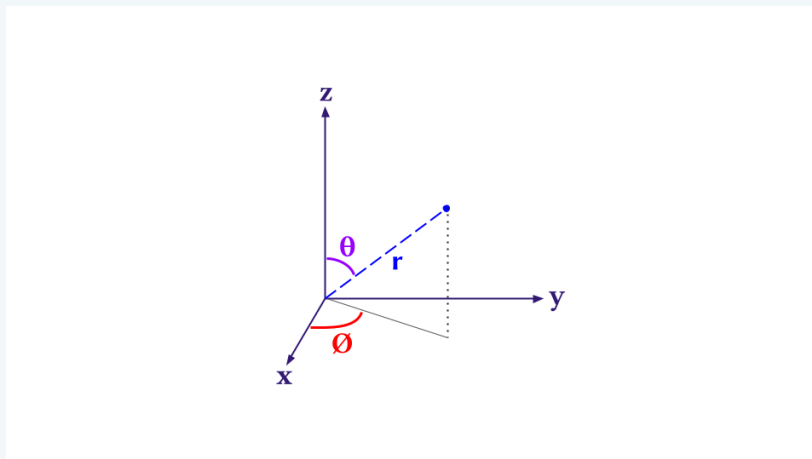


Figure 1: Illustration of r , θ and ϕ in 3D space.

The relation between the Polar and Cartesian coordinates $\{x, y, z\}$ is

$$\begin{aligned} x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta \end{aligned} \quad (2)$$

and $r = \sqrt{x^2 + y^2 + z^2}$. In quantum mechanics the major operator which is of interest to us is the Hamiltonian which comprises of Laplacian or double derivative operator.

How can we transform a Cartesian Laplacian to Spherical?

We need to use multivariable chain rule. The formula for the x component is

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi} \quad (3)$$

It is similar for y and z . Just replace the x with y or z and the equation would be in front of you.

Obtaining $\left\{ \frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z} \right\}$

Using $r = \sqrt{x^2 + y^2 + z^2}$ or, $r^2 = x^2 + y^2 + z^2$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{and} \quad \frac{\partial r}{\partial y} = \frac{y}{r} \quad \text{and} \quad \frac{\partial r}{\partial z} = \frac{z}{r} \quad (4)$$

Substituting the values of $\{x, y, z\}$ in polar form we will easily obtain

$$\begin{aligned}\frac{\partial r}{\partial x} &= \sin \theta \cos \phi \\ \frac{\partial r}{\partial y} &= \sin \theta \sin \phi \\ \frac{\partial r}{\partial z} &= \cos \theta\end{aligned}\tag{5}$$

Obtaining $\left\{\frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial z}\right\}$

The expression with only θ is

$$z = r \cos \theta \Rightarrow \cos \theta = \frac{z}{r}\tag{6}$$

from this we need to derive the partial derivatives of θ wrt cartesian variables, so differentiating the above equation wrt x gives

$$\begin{aligned}-\sin \theta \frac{\partial \theta}{\partial x} &= -\frac{z}{r^2} \frac{\partial r}{\partial x} \\ \Rightarrow \frac{\partial \theta}{\partial x} &= \frac{r \cos \theta}{r^2 \sin \theta} \sin \theta \cos \phi \\ \Rightarrow \frac{\partial \theta}{\partial x} &= \frac{\cos \theta \cos \phi}{r}\end{aligned}\tag{7}$$

In a similar case

$$\begin{aligned}\frac{\partial \theta}{\partial y} &= \frac{\cos \theta \sin \phi}{r} \\ \frac{\partial \theta}{\partial z} &= -\frac{\sin \theta}{r}\end{aligned}\tag{8}$$

Obtaining $\left\{\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right\}$

The expression for ϕ exclusively can be found in dividing y/x

$$\frac{y}{x} = \frac{r \sin \theta \sin \phi}{r \sin \theta \cos \phi} \Rightarrow \tan \phi = \frac{y}{x}\tag{9}$$

Differentiating the above expression wrt x gives

$$\begin{aligned}-\frac{y}{x^2} &= \sec^2 \phi \frac{\partial \phi}{\partial x} \\ \Rightarrow \frac{\partial \phi}{\partial x} &= -\frac{y \cos^2 \phi}{x^2} \\ \Rightarrow \frac{\partial \phi}{\partial x} &= -\frac{r \sin \theta \sin \phi \cos^2 \phi}{r^2 \sin^2 \theta \cos^2 \phi} \\ \therefore \frac{\partial \phi}{\partial x} &= -\frac{\sin \phi}{r \sin \theta}\end{aligned}\tag{10}$$

Similarly

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= \frac{\cos \phi}{r \sin \theta} \\ \frac{\partial \phi}{\partial z} &= 0\end{aligned}\tag{11}$$

Combine and Make $\frac{\partial}{\partial x}$

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial x} &= \sin \theta \cos \phi \frac{\partial}{\partial r} + \frac{\cos \theta \cos \phi}{r} \frac{\partial}{\partial \theta} - \frac{\sin \phi}{r \sin \theta} \frac{\partial}{\partial \phi}\end{aligned}\tag{12}$$

After similar treatment in the y and z we can easily deduce the first order derivatives

$$\frac{\partial}{\partial y} = \sin \theta \sin \phi \frac{\partial}{\partial r} + \frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} + \frac{\cos \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \quad (13)$$

and lastly for z

$$\frac{\partial}{\partial z} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \quad (14)$$

From here on the $\frac{\partial^2}{\partial x^2}$ is much messier because we need to differentiate a differentiated function. Principally, one can do it on pen and paper, but we rather use either wolfram mathematica or sympy from **python**, whatever is available to compute symbolic algebra.

! Important

It is to be noted that Wolfram has been built over decades so it has the richest symbolic algebra system in the world right now. But the only issue is its closedness. Hence, the reachability to many students, I believe, would be low. Hence for those I give the python sympy alternative. I also tried Julia's Symbolics.jl package but it was not so easy for me. In future when I get used to it I will also provide Julia tutorial too for this. For now we stick with Wolfram and Python

Wolfram

If one has Wolfram Mathematica license or open source license then it is much easier to look at Laplacian directly in different coordinate system.

```
Laplacian[f[r, theta, phi], {r, theta, phi}, "Spherical"] //Simplify //Expand
```

The result would be obtained directly and there would be terms like $f^{(2,0,0)}[r, \theta, \phi]$ which is equivalent to $\frac{\partial^2 f}{\partial r^2}$

Python

In sympy one has to do a little bit coding to derive it, first importing the package

i Displaying Formula and Equations

Use Jupyter notebook environment for the latex equation rendering and the function `display(expr)` automatically renders a latex expression in the notebook output cell. If one wants to have raw latex syntax for the expression one should use `latex(expr)` for it. I have not tried marimo notebooks for this stuff, but I hope it would work the same.

```
from sympy import *
```

💡 Tip

If one wants to import like the following

```
import sympy as sp
```

then all the functions need to be called like

```
d_expr = sp.diff(expr)
```

Defining variables and functions

```
r, theta, phi = symbols('r theta phi', positive=True, real=True)
f = Function('f')(r, theta, phi)
```

Defining a python function to differentiate, although one can do it manually hard-coding by using `diff()` sympy function.

```
def diff_xi(cf_r, cf_theta, cf_phi, expr):
    return cf_r * diff(expr, r) + cf_theta * diff(expr, theta) + cf_phi * diff(expr, phi)
```

The `cf_r` and others are coefficients of the derivative expression (*which are derived earlier by hand*). So one needs to put them into variables

```
# x coeffs
ax_r = sin(theta) * cos(phi)
ax_theta = cos(theta) * cos(phi) / r
ax_phi = -sin(phi) / (r * sin(theta))
# y coeffs
ay_r = sin(theta) * sin(phi)
ay_theta = cos(theta) * sin(phi) / r
ay_phi = cos(phi) / (r * sin(theta))
# z coeffs
az_r = cos(theta)
az_theta = -sin(theta) / r
az_phi = 0
```

Do the double derivative using the custom `diff_xi()` function.

```
d2f_dx2 = diff_xi(ax_r, ax_theta, ax_phi, diff_xi(ax_r, ax_theta, ax_phi, f))
d2f_dy2 = diff_xi(ay_r, ay_theta, ay_phi, diff_xi(ay_r, ay_theta, ay_phi, f))
d2f_dz2 = diff_xi(az_r, az_theta, az_phi, diff_xi(az_r, az_theta, az_phi, f))
```

Calculate the Laplacian by summing them all up.

```
laplacian = d2f_dx2 + d2f_dy2 + d2f_dz2
```

Now display the Laplacian

```
display(expand(simplify(laplacian)))
```

The output is

$$\frac{\partial^2}{\partial r^2} f(r, \theta, \phi) + \frac{2}{r} \frac{\partial}{\partial r} f(r, \theta, \phi) + \frac{\partial^2}{\partial \theta^2} f(r, \theta, \phi) + \frac{\partial}{\partial \theta} f(r, \theta, \phi) \frac{1}{r^2 \tan(\theta)} + \frac{\partial^2}{\partial \phi^2} f(r, \theta, \phi) \frac{1}{r^2 \sin^2(\theta)} \quad (15)$$

which can be simplified further to be written as

$$\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \quad (16)$$

The equation can be also written in a way which textbooks write it.

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \quad (17)$$

Cylindrical Coordinate System

Similar to polar coordinate system, cylindrical coordinate system can be visualized as follows where the degree of freedom is $\{\rho, \phi, z\}$.

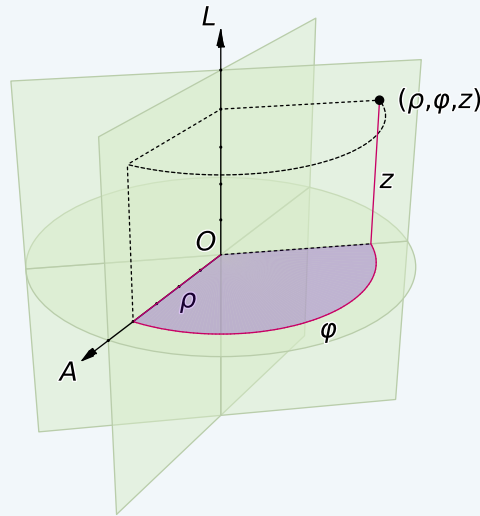


Figure 2: Visualization of Cylindrical coordinate system. Attribution: By Jorge Stolfi - Own work, Public Domain, <https://commons.wikimedia.org/w/index.php?curid=6692609>

One can go through hardships again using pen and paper and derive the Laplacian in cylindrical coordinate system exactly like in the previous case. Or if you are lazy like me use wolfram or use a hybrid method with sympy. Here I use wolfram. Firstly, the relation between the Cartesian and Cylindrical can also be seen using the following command

```
CoordinateTransform["Cylindrical" -> "Cartesian", {rho, phi, z}]
```

The relations are

$$\begin{aligned} x &= \rho \cos \phi \\ y &= \rho \sin \phi \\ z &= z \end{aligned} \quad (18)$$

where

$$\rho = \sqrt{x^2 + y^2} \quad (19)$$

Just like the previous treatment, one can use the same chain rule logic and find the Laplacian or

```
Laplacian[f[rho, phi, z], {rho, phi, z}, "Cylindrical"] //Simplify //Expand
```

and the result is

$$\nabla^2 = \frac{\partial^2}{\partial z^2} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial \rho^2} \quad (20)$$